hep-th/0210223

Thomson scattering of chiral tensors and scalars against a self-dual string

Pär Arvidsson[1], **Erik Flink**[2] **and Måns Henningson**[3]

Department of Theoretical Physics
Chalmers University of Technology and Göteborg University
S-412 96 Göteborg, Sweden

Abstract: We give a non-technical outline of a program to study the $(2,0)$ theories in six space-time dimensions. Away from the origin of their moduli space, these theories describe the interactions of tensor multiplets and self-dual spinning strings. We argue that if the ratio between the square of the energy of a process and the string tension is taken to be small, it should be possible to study the dynamics of such a system perturbatively in this parameter. As a first step in this direction, we perform a classical computation of the amplitude for scattering chiral tensor and scalar fields (i.e. the bosonic part of a tensor multiplet) against a self-dual spinnless string.

[1]par@fy.chalmers.se
[2]erik.flink@fy.chalmers.se
[3]mans@fy.chalmers.se

1 Introduction

One of the most surprising discoveries in string theory during the past decade is the existence of a new class of superconformal quantum theories in six space-time dimensions, that do not contain dynamical gravity. These so called $(2,0)$ theories (the name refers to the supersymmetry algebra under which they are invariant [?]) made their first appearance in the study of type IIB string theory compactified on a four-dimensional hyper-Kähler manifold [?]. If we approach a point in the moduli space of the hyper-Kähler manifold where it develops an isolated singularity, and simultaneously take the string coupling constant to zero, a six-dimensional theory at the locus of the singularity decouples from the ten-dimensional bulk theory. The possible singularities of a hyper-Kähler manifold obey an ADE-classification, i.e. there are two infinite series A_r, $r = 1, 2, \ldots$ and D_r, $r = 3, 4, \ldots$, and three 'exceptional' cases E_6, E_7 and E_8. Each type of singularity gives rise to a different version of $(2,0)$ theory, but these theories have no other discrete or continuous parameters. However, the approach to the singular point in the moduli space of the hyper-Kähler manifold and the possibility to turn on holonomies of the two-form gauge potential of type IIB string theory can be described by $5r$ real moduli, where r is the rank of the singularity.

For the A-series of $(2,0)$ theory, there is an alternative description, in which the A_r theory is regarded as the world-volume theory on $r+1$ parallel $M5$-branes in M-theory [?]. The $5r$ moduli are then given by the relative transverse positions of the $M5$-branes in the eleven-dimensional bulk space. Similarly, one may describe the D-series of $(2,0)$ theory by including also a parallel orientifold plane in the configuration. However, the E_6, E_7 and E_8 versions of $(2,0)$ theory have no known realization within M-theory.

An intrinsically six-dimensional definition of the $(2,0)$ theories, which does not rely on their embedding into a ten- or eleven-dimensional theory, is still lacking, though. In this paper, we will suggest a program to study the $(2,0)$ theories from a purely six-dimensional perspective, with the hope that this work will eventually lead to such a definition. Our approach is based on an analogy between the $(2,0)$ theories and non-abelian maximally supersymmetric Yang-Mills theory in five space-time dimensions. In the next section, we will give a rather non-technical discussion of the degrees of freedom of the $(2,0)$ theories and their relationship to those of Yang-Mills theory. In section three, we will then outline a perturbative scheme to study the quantum dynamics of these degrees of freedom. We intend to systematically explore this in forthcoming publications. As a first step, we will in the present paper describe a purely classical computation of certain scattering amplitudes. We will be working in a rather simplified toy model, which is closely related to the A_1 version of $(2,0)$ theory, but we think that our analysis still captures many aspects of the complete theory correctly.

A different approach to $(2,0)$ theory is described in [?].

2 The degrees of freedom of $(2,0)$ theory

The M-theory realization of the A-series of $(2,0)$ theory provides a convenient starting point for analyzing the degrees of freedom of the theory. Starting with a single $M5$-brane, its fluctuations may be described by a free massless $(2,0)$ tensor multiplet on the brane world-volume. Such a multiplet comprises a real two-form gauge field with self-dual three-form field strength. We will refer to such a field as a 'chiral' two-form gauge field. It is a singlet under the $SO(5) \simeq Sp(4)$ R-symmetry group which is part of $(2,0)$ supersymmetry. There are also five real scalar fields in the vector representation of $SO(5)$, and four chiral spinor fields in the vector representation of $Sp(4)$, that obey a symplectic Majorana condition. Upon quantization, these fields give rise to massless particles, that may be described by their transformation properties under the $SO(4)$ little group that leaves the light-like six-momentum invariant. In this way, the two-form gauge field provides three states in the rank two anti-symmetric self-dual tensor representation of $SO(4)$, and the scalar fields provide five $SO(4)$ invariant states. In total there are thus eight bosonic states. Similarly, the spinor fields provide eight fermionic states, that transform as two $SO(4)$ spinors. With $r+1$ parallel $M5$-branes, as is relevant for the A_r version of $(2,0)$ theory, we have one such tensor multiplet for each $M5$-brane, but one linear combination (namely the sum) decouples from the remaining theory and will henceforth not be considered. (This is analogous to the fact that the world-volume field theory of $r+1$ parallel $D3$-branes in type IIB string theory is a four-dimensional $N=4$ super Yang-Mills theory with gauge group $SU(r+1)$ rather than $U(r+1)$ [?].)

Additional degrees of freedom come from open $M2$-branes that stretch from one of the $r+1$ $M5$-branes to another. From the six-dimensional world-volume perspective, these are perceived as strings, with a tension given by the tension of the $M2$-brane times the transverse distance between the two $M5$-branes in question. At the origin of the moduli space, the $M5$-branes are coincident and the strings are thus tensionless. Interchanging the roles of the two $M5$-branes amounts to changing the orientation of the string. The string is self-dual in the sense that it has equal electric and magnetic charges with respect to the chiral two-form gauge fields that are part of the tensor multiplets on the two $M5$-branes. (These charges obey Dirac quantization, and thus cannot be taken to be small.) In the presence of such a string, the $(2,0)$ supersymmetry algebra develops a 'central' charge, which is a vector under the $SO(5,1)$ Lorentz group and also a vector under the $SO(5)$ R-symmetry group [?]. The representation theory of the $(2,0)$ supersymmetry algebra with such a central charge reveals that the string transforms non-trivially both under rotations in the space transverse to the string and under the subgroup of the $SO(5)$ R-symmetry group that is unbroken by the modulus corresponding to the distance between the $M5$-branes. A minimal 'BPS-saturated' representation corresponds to a multiplet of straight strings [?], while non BPS-saturated representations describe waves propagating along the strings. These degrees of freedom can be realized by complementing the scalar fields on the string world-sheet, which represent the transverse space-time coordinates of the string, with a set of fermionic fields.

We now turn to the realization of $(2,0)$ theory within type IIB string theory compactified on a hyper-Kähler manifold that develops an ADE-singularity. Zero-modes of the massless type IIB fields will then give rise to r tensor multiplets, where r is the rank of the singularity. Furthermore, $D3$-branes that wrap around homologically non-trivial two-spheres of the hyper-Kähler manifold will be perceived as strings in six dimensions. There is in fact such a two-sphere for each positive root of the corresponding ADE-type Lie algebra (i.e. $su(r+1)$ for A_r, $so(2r)$ for D_r, and the E_6, E_7 and E_8 Lie algebras.) The tension of such a string is given by the tension of the $D3$-brane times the area of the two-sphere. The latter is given by the values of the moduli, and vanishes at the origin of the moduli space, where the strings thus become tensionless. In summary, we have been led to the following picture of the degrees of freedom of $(2,0)$ theory: With each Cartan generator of the corresponding ADE-type Lie algebra is associated a tensor multiplet of particles. With each positive root generator of the Lie algebra is associated a multiplet of straight strings. Finally, there are excitations that describe propagating waves along these strings.

In six dimensions, the degrees of freedom associated with the Cartan generators and the root generators of the ADE-type Lie algebra are rather different, and no Lie group symmetry is apparent in the theory. However, if we compactify the theory on a circle, the tensor multiplets associated with the Cartan generators give rise to massless vector multiplets of maximally extended supersymmetry in five dimensions. On the other hand, the string multiplets associated with the root generators give rise to massive vector multiplets, if the strings are straight and wound around the circle. The mass of these multiplets is given by the tension of the string times the radius of the circle. Strings that are not wound around the circle appear as magnetically charged strings in five dimensions. We thus recover the degrees of freedom of maximally supersymmetric Yang-Mills theory, with an ADE-type gauge group spontaneously broken down to the maximal torus group by the vacuum expectation values of the moduli fields. As we approach the origin of the moduli space, the massive vector multiplets are un-Higgsed and also become massless vector multiplets. The non-abelian gauge-symmetry of the theory then becomes apparent. In six dimensions, this transition possibly corresponds to a theory of tensionless strongly interacting self-dual strings, but we do not yet have a clear picture of what this would mean. In particular, gauge invariance plays a crucial role for the consistency of the five-dimensional Yang-Mills theory, and should derive from some more general principle in six dimensions, but exactly how this comes about is still rather unclear. In this way, we may regard the compactified $(2,0)$ theory as the ultraviolet completion needed to make five-dimensional maximally supersymmetric Yang-Mills theory into a consistent theory.

3 Investigating the quantum dynamics

Their high degree of uniqueness is of course an attractive feature of the $(2, 0)$ theories, but at the same time makes their study more difficult. In particular, one might think that the absence of a small parameter would make a perturbative analysis impossible, and force us to define and solve the dynamics of the theory exactly. Fortunately, this is not the case, as we will now explain: The approach that we would like to advocate is to consider the theory at a point away from the origin of the moduli space, so that the strings associated with the root generators of the Lie algebra are tensile. (In any case, our understanding of the degrees of freedom at the origin of the moduli space is still too limited to allow us to work there.) Once the vacuum expectation values of the moduli are fixed, a state in the theory can be characterized by giving the number of infinitely extended approximately straight strings, together with their spatial directions and momenta. In addition, the vibrational state of the string, which describes waves propagating along the string, should be specified. Finally, the state may contain a number of tensor multiplet particle quanta of various types.

The energy of such a state is of course in general infinite, because of the mass of the infinitely extended strings. However, since only energy differences between different states affect the dynamics of the theory, a more relevant statement is that the energy difference between two states containing different numbers of strings is infinite. This means that the Hilbert space of the theory decomposes into separate superselection sectors, characterized by the number of strings and their spatial directions. (This conclusion would of course not hold if the theory is compactified on a circle, so that straight strings have finite energy. Indeed, while charge is conserved in Yang-Mills theory, particle number is not.) In each superselection sector there is a ground state, containing only straight strings with certain spatial directions and momenta. Above this ground state, there are excited states that also contain quanta of propagating waves along the strings and tensor multiplet particles. The energies of these excitations only depend on the corresponding wavelengths and not on the string tension. Furthermore, in the limit where the string tension becomes large compared to the energies of these excitations, the excitations decouple from each other. We are then left with a theory of free tensor multiplets and free waves propagating on the strings, i.e. an infinite set of free harmonic oscillators. The latter theory is of course exactly solvable, and can serve as a starting point for a perturbative analysis. The parameter of such a perturbation expansion is given by the square of the energy of a process (e.g. the energy of incoming tensor multiplet quanta), divided by the string tension.

Sofar, we have only discussed open infinitely extended strings. One might also worry about the possible influence of closed strings. However, on dimensional grounds, we would expect the quantization of such strings to yield states with an energy of the order of the square root of the string tension, so they can be safely neglected in the limit that the tension goes to infinity. (If there are any massless states in the spectrum of closed strings, other than the tensor multiplets that we have been discussing, we expect them to decouple from the rest of the theory.)

In view of the above considerations, it is natural to start exploring $(2,0)$ theory with a small number of infinitely extended strings. Qualitatively new features then arise as this number is increased. With no strings involved, we have the quantum theory of free tensor multiplets, which is by now well understood. With a single string, one could investigate the scattering of tensor multiplet quanta against it. With two strings, the interaction of the strings with each other can be studied. By analogy with Yang-Mills theory, which in addition to the trilinear couplings contains also quadrilinear couplings, one would expect that in addition to interactions mediated by the exchange of tensor multiplet quanta, there should also be direct string-string interactions. (On the other hand, we do not expect any new couplings to appear as the number of strings is increased further.) An additional subtlety is related to the electric and magnetic charges of the strings. They imply that the quantum wave function obeys Dirac quantization, i.e. it is a section of a line bundle rather than a function over the configuration space of the strings.

In a longer perspective, one would of course wish for a 'second quantized' formalism, capable of describing an arbitrary number of strings. Presumably, this is best formulated in an expansion around the origin of the moduli space, where the strings are tensionless. As remarked above, such a formulation becomes necessary in order to describe compactifications of $(2,0)$ theory, in which straight strings can have a finite mass and sectors with different number of strings mix with eachother. As in many other examples in string and M-theory, the dynamics of $(2,0)$ theory is thus considerably richer in a compactified space-time.

4 The classical limit

In our future work, we intend to systematically explore the quantum dynamics of $(2,0)$ theory as outlined above. However, before embarking on this program, it is natural to ask if there are any questions that can be answered by a purely classical calculation.

First of all, we would expect such a model to be valid only in the extreme low-energy limit, where the tensor multiplets can be treated as classical fields while neglecting their quantum properties. Furthermore, because of Dirac quantization effects, we will not be able to properly investigate states with more than one string. Finally, we will be limited to studying the bosonic degrees of freedom, i.e. we can only consider the chiral two-form gauge fields and the scalar fields of the tensor multiplets, and the string will only have its transverse space-time coordinates as fields on the world-sheet (i.e. there are no fermionic fields on the world-sheet).

4.1 The model

For simplicity, we will work with (the bosonic part of) a single tensor multiplet and a single type of (spinnless) string, i.e. a simplified theory related to the A_1-version of $(2,0)$ theory (which reduces to $SU(2)$ Yang-Mills theory upon compactification). The fields are thus locally a chiral two-form B and five scalar fields ϕ^A, $A = 1, \ldots, 5$ in a $(5+1)$-dimensional Minkowski space-time with signature $(-+++++)$. The values of the scalar fields at spatial infinity ϕ^A_∞ constitute the moduli of the model. We may define these fields so that $\phi^2_\infty = \ldots = \phi^5_\infty = 0$. In fact, the fields $\phi^2, \ldots, \phi^5$ then decouple completely from the rest of the theory, and henceforth we will only consider a single scalar field $\phi = \phi^1$. (This would not be so in the complete $(2,0)$ theory, where fermionic degrees of freedom are included.) The string is described by its two-dimensional world-sheet Σ, which we parametrize with some coordinates τ and σ. Its embedding into space-time is given by a set of functions $X^\mu(\tau, \sigma)$, $\mu = 0, 1, \ldots, 5$.

We wish to describe the dynamics of these degrees of freedom in a Lagrangian formalism. As is well known, this poses a problem for the chiral two-form B, which does not admit such a formulation [**?**]. We therefore relax the condition that the three-form field strength H of B be self-dual. However, our model will have the property that the 'anti-chiral' part of B, i.e. the part with an anti self-dual field strength, is a 'spectator field' that decouples completely from the other degrees of freedom. We then define the action of the model as

$$S = -\frac{1}{4\pi}\int H \wedge *H - \frac{1}{8\pi}\int d\phi \wedge *d\phi - \int_\Sigma B + \int_\Sigma \phi \mathrm{Vol}_\Sigma. \tag{1}$$

The first term is the generalized Maxwell action for the two-form gauge-field B, while the second is the kinetic term of the scalar field ϕ. The third term incorporates the electric coupling between the string and the gauge field, and the fourth term is the Nambu-Goto term for the string. Here, Vol_Σ is the volume form on the world-sheet Σ induced by its embedding into six-dimensional Minkowski space, i.e.

$$\mathrm{Vol}_\Sigma \equiv d\tau \wedge d\sigma \sqrt{-\mathrm{det} g_{\alpha\beta}} = d\tau \wedge d\sigma \sqrt{-\mathrm{det}(\partial_\alpha X^\mu \partial_\beta X_\mu)}, \tag{2}$$

where $g_{\alpha\beta}$ is the induced metric on the world-sheet. The indices α and β take the values 0 and 1, corresponding to τ and σ, respectively. The string tension is thus formally given by the negative of the value of the scalar field ϕ at the locus of the string world-sheet Σ.

In addition to its electric coupling, the self-dual string should also have a magnetic coupling, which we put in 'by hand' [**?**, **?**] by defining the field strength H as

$$H = dB + H_\Sigma. \tag{3}$$

Here, H_Σ is a three-form that only depends on the embedding coordinates X^μ of Σ, and obeys the 'Bianchi identity'

$$dH_\Sigma = 2\pi\delta_\Sigma. \tag{4}$$

The four-form δ_Σ is the 'Poincaré dual' of the world-sheet Σ, in the sense that

$$\int_\Sigma s \equiv \int \delta_\Sigma \wedge s \tag{5}$$

for an arbitrary two-form s over Minkowski space. (Like in many other formulas in this paper, we do not here notationally distinguish a form over Minkowski space from its pullback to the world-sheet by the embedding functions X^μ.)

There are in fact no parameters that can be continuously adjusted in this action. The numerical constant preceding the first term of the action is fixed by requiring that the anti self-dual part of H decouples, while the constant in front of the second term is determined by requiring that the net force between two infinitely long and parallel straight strings vanishes, as we will see in the end of this section. The constant multiplying the electric coupling term is fixed by demanding invariance under 'large' gauge transformations of the two-form gauge-field B, whereas a constant multiplying the Nambu-Goto term can be absorbed in a redefinition of the scalar field ϕ.

In the remainder of this section, we first derive the equations of motion for the string and the fields. We then construct a configuration describing a static, infinitely long, straight string. Finally, we check that two parallel such strings do not exert any forces on eachother.

4.2 Equations of motion

The equations of motion are obtained by requiring the action (1) to be stationary under variations of the generalized coordinates, i.e. the gauge field B, the scalar field ϕ, and the coordinates X^μ that describe the embedding of the string world-sheet in Minkowski space. Because of the invariance of the action under reparametrizations of the world-sheet, we may impose two independent conditions on the X^μ, though. A convenient choice for our purposes is to parametrize the world-sheet by $\tau = X^0 = x^0$ and $\sigma = X^5 = x^5$. The remaining X^i, $i = 1, \ldots, 4$, then describe the fluctuations around a static configuration containing an infinitely long, straight string in the x^5-direction. This parametrization may obviously break down for large fluctuations, where the X^i are no longer necessarily single-valued functions of x^0 and x^5.

We now start by varying the action with respect to B, and obtain the equations of motion

$$d^*H = 2\pi * \delta_\Sigma, \tag{6}$$

which is consistent with the Bianchi identity (4) and the self-duality condition

$$H = *H, \tag{7}$$

as was required. This fixes the numerical coefficient of the first term in Eq. (1).

Next, we vary the action with respect to ϕ, and obtain the equation of motion

$$d^* d\phi = -4\pi * (\delta_\Sigma \wedge \widehat{\mathrm{Vol}}_\Sigma), \tag{8}$$

where $\widehat{\mathrm{Vol}}_\Sigma$ is some two-form over Minkowski space whose pull-back to the world-sheet equals Vol_Σ.

Finally, it remains to vary the action with respect to the string coordinates X^μ. This is a lot more involved than the previous calculations, therefore we will be more explicit here. The reader may wish to skip this rather technical derivation and jump directly to Eq. (27).

To determine the variation of the first term in Eq. (1), we first note that only H_Σ, and not B, depends on X. From the Bianchi identity (4) follows that the variation of H_Σ is of the form

$$\delta H_\Sigma = 2\pi\omega + d\Lambda, \tag{9}$$

where Λ is some two-form, and the three-form ω is related to the variation of the Poincaré dual δ_Σ as

$$d\omega = \delta\delta_\Sigma. \tag{10}$$

Thus, we get that

$$\delta\left(-\frac{1}{4\pi}\int H \wedge *H\right) = -\frac{1}{2\pi}\int H \wedge *\delta H_\Sigma = -\int \omega \wedge *H + \int_\Sigma \Lambda, \tag{11}$$

where we also used the equations of motion (6) for H. To proceed we must determine the three-form ω and the two-form Λ. The covariant expression for δ_Σ is

$$\delta_\Sigma = -\int d\tau d\sigma \delta^{(6)}(x - X(\tau,\sigma))\dot{X}_\mu \acute{X}_\nu * (dx^\mu \wedge dx^\nu), \tag{12}$$

which is easily verified by insertion in Eq. (5). Here a dot denotes a derivative with respect to τ while a prime denotes a σ-derivative. From the variation of this quantity and Eq. (10), we see that we can take

$$\omega = -\int d\tau d\sigma \delta^{(6)}(x - X(\tau,\sigma))\delta X_\mu \dot{X}_\nu \acute{X}_\rho * (dx^\mu \wedge dx^\nu \wedge dx^\rho), \tag{13}$$

where we have used the contraction identities for Levi-Civita symbols.

To find Λ, we need some knowledge of the particular solution H_Σ to Eq. (4). This, however, is not possible to obtain in a simple covariant form. With our choice of parametrization of the world-sheet, as described in the beginning of this subsection, we can write δ_Σ as

$$\begin{aligned}\delta_\Sigma \;=\; & \frac{\epsilon_{ijkl}}{4!}\delta^{(4)}(x - X)\Big(dx^i \wedge dx^j \wedge dx^k \wedge dx^l - 4\dot{X}^i dx^0 \wedge dx^j \wedge dx^k \wedge dx^l \\ & - 4\acute{X}^i dx^5 \wedge dx^j \wedge dx^k \wedge dx^l + 12\dot{X}^i \acute{X}^j dx^0 \wedge dx^5 \wedge dx^k \wedge dx^l\Big).\end{aligned} \tag{14}$$

Here the dot and the prime denote x^0 and x^5 derivatives respectively, and the four-index Levi-Civita symbol is defined by $\epsilon^{i_1 i_2 i_3 i_4} \equiv \epsilon^{0 i_1 i_2 i_3 i_4 5}$. The first term describes a static, straight string while the following terms allow for changes in time and for bending of the string. A solution to the Bianchi identity (4) is then

$$\begin{aligned} H_\Sigma &= \frac{1}{3!} H^\Sigma_{ijk} \Big(dx^i \wedge dx^j \wedge dx^k - 3\dot{X}^i dx^0 \wedge dx^j \wedge dx^k - \\ &\quad - 3\acute{X}^i dx^5 \wedge dx^j \wedge dx^k + 6\dot{X}^i \acute{X}^j dx^0 \wedge dx^5 \wedge dx^k \Big), \end{aligned} \tag{15}$$

where H^Σ_{ijk} fulfills

$$\partial_{[l} H^\Sigma_{ijk]} = \frac{\pi}{2} \epsilon_{lijk} \delta^{(4)}(x - X(\tau, \sigma)). \tag{16}$$

This solution is of course unique only modulo the addition of a closed three-form. Performing a variation of this H_Σ, we find that we can take Λ to be given by

$$\Lambda = -\frac{1}{2} H^\Sigma_{ijk} \delta X^i \Big(dx^j \wedge dx^k - 2\dot{X}^j dx^0 \wedge dx^k - 2\acute{X}^j dx^5 \wedge dx^k + 2\dot{X}^j \acute{X}^k dx^0 \wedge dx^5 \Big). \tag{17}$$

This calculation also yields the expression

$$\begin{aligned} \omega &= \frac{1}{6} \epsilon_{ijkl} \delta^{(4)}(x - X) \delta X^l \Big(dx^i \wedge dx^j \wedge dx^k - 3\dot{X}^i dx^0 \wedge dx^j \wedge dx^k \\ &\quad - 3\acute{X}^i dx^5 \wedge dx^j \wedge dx^k + 6\dot{X}^i \acute{X}^j dx^0 \wedge dx^5 \wedge dx^k \Big), \end{aligned} \tag{18}$$

in agreement with Eq. (13). We can then write the two terms on the right hand side of Eq. (11) in covariant form as

$$\begin{aligned} -\int \omega \wedge *H &= -\int d^6 x \delta^{(4)}(x - X) (*H)_{\mu\nu\rho} \delta X^\mu \dot{X}^\nu \acute{X}^\rho \\ &= -\int_\Sigma d\tau d\sigma (*H)_{\mu\nu\rho} \delta X^\mu \dot{X}^\nu \acute{X}^\rho \end{aligned} \tag{19}$$

and

$$\int_\Sigma \Lambda = -\int_\Sigma d\tau d\sigma H^\Sigma_{\mu\nu\rho} \delta X^\mu \dot{X}^\nu \acute{X}^\rho. \tag{20}$$

The variation of the electric coupling term is

$$\delta \left(-\int_\Sigma B \right) = \delta \left(-\int_\Sigma d\tau d\sigma B_{\mu\nu} \dot{X}^\mu \acute{X}^\nu \right), \tag{21}$$

which after a partial integrations becomes

$$\delta \left(-\int_\Sigma B \right) = -\int_\Sigma d\tau d\sigma (dB)_{\mu\nu\rho} \delta X^\mu \dot{X}^\nu \acute{X}^\rho. \tag{22}$$

We see that this term combines with the one obtained in Eq. (20) into a single term involving $H = H_\Sigma + dB$. Thus, we have

$$\delta\left(-\frac{1}{4\pi}\int H \wedge *H - \int_\Sigma B\right) = -\int_\Sigma d\tau d\sigma (H + *H)_{\mu\nu\rho}\delta X^\mu \dot{X}^\nu \acute{X}^\rho, \tag{23}$$

which shows the symmetry between the electric and magnetic couplings explicitly. We see that an anti self-dual part of the field strength H would cancel from the variation.

Finally, the variation of the Nambu-Goto term is given by

$$\delta \int d\tau d\sigma \phi \sqrt{-\det g_{\alpha\beta}} = \int d\tau d\sigma \sqrt{-\det g_{\alpha\beta}} (\delta X^\mu \partial_\mu \phi + \phi \frac{1}{2} g^{\alpha\beta} \delta g_{\alpha\beta}). \tag{24}$$

Here, $g^{\alpha\beta}$ is the inverse of the induced world-sheet metric $g_{\alpha\beta}$. Inserting that

$$\delta g_{\alpha\beta} = \partial_\alpha \delta X^\mu \partial_\beta X_\mu + \partial_\alpha X^\mu \partial_\beta \delta X_\mu \tag{25}$$

and performing a partial integration of the second term in Eq. (24), we find that this equals

$$\int d\tau d\sigma \sqrt{-\det g_{\alpha\beta}} \delta X^\mu \left(\partial_\mu \phi - \partial_\nu \phi \partial^\alpha X_\mu \partial_\alpha X^\nu - \phi D^\alpha D_\alpha X_\mu\right), \tag{26}$$

where D^α is the covariant world-sheet derivative. One should note that the X^μ, while being a Minkowski space vector, are world-sheet scalars.

Altogether, we find that the equations of motion that follow from varying X^μ can be written as

$$\phi D^\alpha D_\alpha X_\mu = \partial_\mu \phi - \partial_\nu \phi \partial^\alpha X_\mu \partial_\alpha X^\nu - (H + *H)_{\mu\nu\rho} \dot{X}^\nu \acute{X}^\rho, \tag{27}$$

evaluated at the locus of the string, i.e. at $x^\mu = X^\mu$.

4.3 Static solutions

We will now construct a configuration corresponding to an infinitely extended static straight string along the x^5-direction located at $x^i = 0$, $i = 1, \ldots, 4$, i.e. with $X^i = 0$. For such a string, we have

$$\delta_\Sigma = \frac{\epsilon_{ijkl}}{4!} \delta^{(4)}(x) dx^i \wedge dx^j \wedge dx^k \wedge dx^l. \tag{28}$$

The H and ϕ fields are given by

$$H = \frac{1}{\pi} \frac{x_l}{|x|^4} \left(\frac{1}{3!} \epsilon^l{}_{ijk} dx^i \wedge dx^j \wedge dx^k + dx^0 \wedge dx^l \wedge dx^5\right) \tag{29}$$

and

$$\phi = \phi_\infty + \frac{1}{\pi} \frac{1}{|x|^2}. \tag{30}$$

Here $|x| = \sqrt{x^i x_i}$, and ϕ_∞ is an arbitrary constant, that gives the value of ϕ at spatial infinity. It is straight-forward to verify that this configuration satisfies the self-duality condition (7) and the equations of motion (6), (8), and (27). (By rotational invariance in the transverse space, the right-hand side of the latter equation must vanish at $x^i = 0$.)

Since the string is infinitely long, we expect that the energy of this configuration diverges. It is however interesting to consider the energy per unit length along the x^5-direction. To this end, we start by determining the energy-momentum tensor for the fields ϕ and H. It is most conveniently defined by coupling the theory to some arbitrary six-dimensional metric $G_{\mu\nu}$ and computing the variation of the action as this metric is varied. With

$$\begin{aligned} S_0 &= -\frac{1}{8\pi}\int d^6x\sqrt{-\det G}G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\ &= -\frac{1}{4\pi}\frac{1}{3!}\int d^6x\sqrt{-\det G}G^{\mu\mu'}G^{\nu\nu'}G^{\rho\rho'}H_{\mu\nu\rho}H_{\mu'\nu'\rho'} \end{aligned} \tag{31}$$

we get that

$$\delta S_0 = \frac{1}{2}\int d^6x\sqrt{-\det G}T^{\mu\nu}\delta G_{\mu\nu}, \tag{32}$$

where the coefficient $T^{\mu\nu}$ is defined as the energy-momentum tensor of the system. Computing $T^{\mu\nu}$ in this way for our model, and then imposing the self-duality condition on H, we arrive at

$$T^{\mu\nu} = \frac{1}{4\pi}\left(\partial^\mu\phi\partial^\nu\phi - \frac{1}{2}G^{\mu\nu}\partial^\rho\phi\partial_\rho\phi\right) + \frac{1}{4\pi}H^{\mu\rho\sigma}H^\nu{}_{\rho\sigma}. \tag{33}$$

The energy per unit length of our configuration is now given by the integral of the T^{00} component over the transverse space parametrized by the x^i plus the tension of string, i.e. the negative of the value of ϕ at the locus of the string. Both of these contributions diverge, though. We can regularize them by replacing the infinitesimally thin string at $x^i = 0$ with a hollow cylinder of radius $\epsilon > 0$. This amounts to integrating T^{00} over the domain $|x| > \epsilon$ and evaluating ϕ at $|x| = \epsilon$. One then finds that the divergences as $\epsilon \to 0$ cancel, and we are left with an effective string tension T, i.e. energy per unit length, that equals $-\phi_\infty$.

Finally, we point out that we may construct a static solution describing parallel strings by a superposition of translations of the solution we have just described. The equilibrium of such a configuration follows from the fact that the right-hand side of Eq. (27), i.e. the force exerted by the fields on a test string, vanishes identically everywhere for the single string solution we have described. Similar results are obtained in [**?**].

5 Scattering from an infinitely long string

The analysis that we will perform is largely analogous to the classical Thomson scattering, in which electromagnetic radiation is scattered against a spinnless massive particle. (See e.g. chapter 14 of [?].) This calculation is usually regarded to be valid to lowest order in the electric charge of the particle, but to be exact as a function of the mass of the particle. We will instead be considering the scattering of a chiral two-form gauge field and scalar field radiation against a self-dual string. Because of the self-duality, the electric and magnetic charges of the string cannot be taken to be small, so our calculation has to be exact in them. Instead, we will take the square of the frequency of the radiation to be small compared to the tension of the string, and work to leading order in this ratio.

In this section, we will first describe the different kinds of plane waves of radiation. We will then determine the response of a string to such an incoming wave. The next step is to compute the reradiation generated by the string motion. Finally, we analyze this outgoing radiation far away from the string, where it appears as another plane wave.

5.1 Plane waves in six dimensions

A plane wave is typically written as $\mathrm{Re}(Ae^{ik_\mu x^\mu})$ and is specified by its (possibly complex) amplitude A and its wave vector k_μ. The modulus of the zeroth component of the wave vector is the energy E of the wave. Our fields are massless and therefore k_μ is a light-like vector, thus $k_\mu k^\mu = 0$.

The ϕ-field in our setup is a scalar field and therefore easily described as (we omit the Re-operators and keep in mind that we should always take the real part of the final answer)

$$\phi = \phi_0 e^{ik_\mu x^\mu}. \tag{34}$$

The B-field, however, is more complicated. Generally, it can be written as

$$B_{\mu\nu} = b_{\mu\nu} e^{ik_\rho x^\rho}, \tag{35}$$

but there are certain conditions on the fifteen components of $b_{\mu\nu}$. First, we choose a Lorentz-type gauge fixing condition

$$\partial^\mu B_{\mu\nu} = 0, \tag{36}$$

which reduces to

$$k^\mu b_{\mu\nu} = 0, \tag{37}$$

for a plane wave. However, this condition does not fix the gauge entirely, since we may perform a transformation

$$b_{\mu\nu} \to b_{\mu\nu} + k_\mu \Lambda_\nu - k_\nu \Lambda_\mu \tag{38}$$

without altering $k^\mu b_{\mu\nu}$, if $k^\mu \Lambda_\mu = 0$. We also see that $\Lambda_\mu = k_\mu$ is a trivial transformation that does not alter $b_{\mu\nu}$. These observations reduce the number of independent components in Λ_μ from six to four, which means that we need four additional conditions to fix the gauge entirely. We may take these to be

$$b_{0a} = 0, \tag{39}$$

where a ranges over the spatial indices, i.e. from 1 to 5. (This yields four independent conditions since $k^a b_{a0} = 0$ from Eq. (37).)

This gauge fixing leaves us with six independent components in $b_{\mu\nu}$. We must also impose a self-duality condition which reduces this number to three, which are interpreted as the polarizations of the gauge field. The self-duality condition reads

$$H_{\mu\nu\rho} = \frac{1}{3!}\epsilon_{\mu\nu\rho}{}^{\sigma\kappa\lambda} H_{\sigma\kappa\lambda} \tag{40}$$

in terms of the field strength H. For a plane wave B, such that $H = dB$, this becomes

$$k_\mu B_{\nu\rho} + k_\nu B_{\rho\mu} + k_\rho B_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho}{}^{\sigma\kappa\lambda} k_\sigma B_{\kappa\lambda}, \tag{41}$$

which immediately gives that

$$B_{ab} = \frac{1}{2}\epsilon_{ab}{}^{cde} B_{cd} \hat{k}_e, \tag{42}$$

where the hat on k_e indicates that it is normalized to have unit modulus. This may be interpreted by saying that B is self-dual in the four-dimensional space orthogonal to k^a.

Now, consider the energy-momentum tensor for these plane waves. By applying Eq. (33) we get that

$$T^{\mu\nu} = \frac{1}{4\pi}\Big[\mathrm{Re}\left(\phi_0 i k^\mu e^{ik_\tau x^\tau}\right)\mathrm{Re}\left(\phi_0 i k^\nu e^{ik_\tau x^\tau}\right) + \mathrm{Re}\left(ik^\mu b_{\rho\sigma} e^{ik_\tau x^\tau}\right)\mathrm{Re}\left(ik^\nu b^{\rho\sigma} e^{ik_\tau x^\tau}\right)\Big] \tag{43}$$

where we have used that $k^\mu k_\mu = 0$. Note the correspondence between $b_{\rho\sigma} b^{\rho\sigma}$ and ϕ_0^2 in this expression, meaning that we have the same normalization with respect to energy for the two types of waves.

5.2 String vibrations

Now that we have explored the area of plane waves, we turn to the actual scattering situation. The philosophy goes as follows: We start from the static case considered previously but add an incident plane wave. This wave will cause the string to vibrate, but the deviations from the static case will be small if the string tension $-\phi_\infty$ is large.

The string vibrations will in turn cause emission of new waves, which however will be much smaller in amplitude than the incident plane wave. This means that we may neglect back reaction, i.e. the action of the outgoing waves on the string.

The general equations of motion for the fields $H(x)$ and $\phi(x)$ and for $X(x^0, x^5)$ are given by Eqs. (6), (8) and (27). They are

$$\begin{aligned} dH &= 2\pi\delta_\Sigma && (44)\\ d^*d\phi &= -4\pi\Omega_\Sigma && (45)\\ \phi(\ddot{X}_i - \acute{\acute{X}}_i) &= -\partial_i\phi + 2(H_{i05} + H_{ij5}\dot{X}^j + H_{i0j}\acute{X}^j) + \mathcal{O}(X^2) && (46) \end{aligned}$$

where we have used that $H = *H$ and introduced $\Omega_\Sigma \equiv *(\delta_\Sigma \wedge \widehat{\mathrm{Vol}}_\Sigma)$. Anticipating that the string fluctuations X^i will be small, we have only expanded the last equation to second order in these.

Since the outgoing waves are suppressed by a factor $\frac{1}{\phi_\infty}$, which is small when the string tension approaches infinity, we may expand these equations in powers of $\frac{\varepsilon^2}{\phi_\infty}$, where ε is some energy scale introduced to get a dimensionless expansion parameter. This gives

$$\begin{aligned} H &= H_{\mathrm{st}} + H_{\mathrm{in}} + \frac{\varepsilon^2}{\phi_\infty}H_{\mathrm{out}} + \mathcal{O}\left(\frac{\varepsilon^4}{\phi_\infty^2}\right) && (47)\\ \phi &= \phi_\infty + \phi_{\mathrm{st}} + \phi_{\mathrm{in}} + \frac{\varepsilon^2}{\phi_\infty}\phi_{\mathrm{out}} + \mathcal{O}\left(\frac{\varepsilon^4}{\phi_\infty^2}\right) && (48)\\ X^i &= 0 + \frac{\varepsilon^2}{\phi_\infty}X^i_{\mathrm{Th}} + \mathcal{O}\left(\frac{\varepsilon^4}{\phi_\infty^2}\right) && (49)\\ \delta_\Sigma &= \delta_{\mathrm{st}} + \frac{\varepsilon^2}{\phi_\infty}\delta_{\mathrm{Th}} + \mathcal{O}\left(\frac{\varepsilon^4}{\phi_\infty^2}\right) && (50)\\ \Omega_\Sigma &= \Omega_{\mathrm{st}} + \frac{\varepsilon^2}{\phi_\infty}\Omega_{\mathrm{Th}} + \mathcal{O}\left(\frac{\varepsilon^4}{\phi_\infty^2}\right). && (51) \end{aligned}$$

Here "st" refers to the static configuration described in subsection 4.3. (ϕ_{st} stands for the static ϕ-field minus the constant ϕ_∞.) H_{in} and ϕ_{in} denote the incoming plane waves. Quantities denoted "Th" (for Thomson) describe the response of the string, whereas reradiation is described by H_{out} and ϕ_{out}. Note that we have explicitly extracted the appropriate powers of $\frac{\varepsilon^2}{\phi_\infty}$.

This allows us to rewrite the equations of motion order by order in $\frac{\varepsilon^2}{\phi_\infty}$. Observing that the static fields must obey the equations of motion for a static, straight string, namely

$$\begin{aligned} dH_{\mathrm{st}} &= 2\pi\delta_{\mathrm{st}} && (52)\\ d^*d\phi_{\mathrm{st}} &= -4\pi\Omega_{\mathrm{st}} && (53)\\ \partial^i\phi_{\mathrm{st}} + 2H^{i05}_{\mathrm{st}} &= 0, && (54) \end{aligned}$$

and that $H_{\rm st}^{ij5} = H_{\rm st}^{i0j} = 0$, we get the following equations:

$$\begin{aligned} dH_{\rm in} &= 0 & (55)\\ d^* d\phi_{\rm in} &= 0 & (56)\\ dH_{\rm out} &= 2\pi\delta_{\rm Th} & (57)\\ d^* d\phi_{\rm out} &= -4\pi\Omega_{\rm Th} & (58)\\ \varepsilon^2(\ddot{X}^i_{\rm Th} - \overset{\prime\prime}{X}{}^i_{\rm Th}) &= -\partial^i\phi_{\rm in} - 2H^{i05}_{\rm in}. & (59)\end{aligned}$$

We see that the outgoing waves do not appear in the equation for $X^i_{\rm Th}$, which allows us to solve for the string vibrations directly, as was anticipated. To solve Eq. (55), we introduce a gauge field $B_{\rm in}$ such that $H_{\rm in} = dB_{\rm in}$. We take this to be a plane wave, satisfying the gauge fixing conditions (36) and (39).

The next step is to solve Eq. (59) for $X^i_{\rm Th}$. We start with the simplest case where $H_{\rm in} = 0$ and $\phi_{\rm in} = \phi_0 e^{ik_\mu x^\mu}$ with the wave vector $k_\mu = -E(1, 0, 0, 0, \sin\theta, \cos\theta)$. The situation is depicted in Fig. 1.

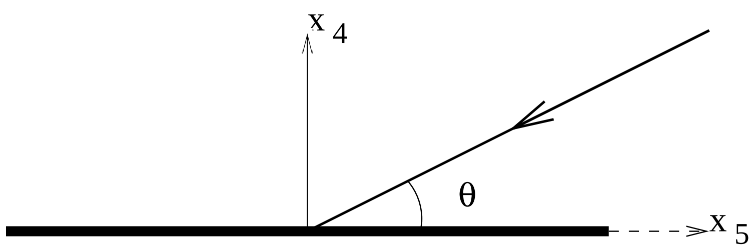

Figure 1: Plane wave incident on an infinitely long string in the x^5-direction.

For this incident plane wave, we get that

$$\partial^i\phi_{\rm in} = -iE\sin\theta\delta^i{}_4\phi_0 e^{-iE(x^0 + x^4\sin\theta + x^5\cos\theta)}, \tag{60}$$

which when inserted in Eq. (59) yields

$$\varepsilon^2(\ddot{X}^i_{\rm Th} - \overset{\prime\prime}{X}{}^i_{\rm Th}) = iE\sin\theta\delta^i{}_4\phi_0 e^{-iE(x^0 + x^5\cos\theta)}, \tag{61}$$

since we may, to first order, let $X_i = 0$ in the exponent. Remember that Eq. (59) should be evaluated on the world-sheet, which in our parametrization is equivalent to letting $x^i = X^i$.

This equation is solved by making the ansatz

$$X^i_{\rm Th} = \xi^i e^{-iE(x^0 + x^5\cos\theta)}, \tag{62}$$

which immediately gives that

$$X^i_{\rm Th} = -\frac{i\phi_0}{\varepsilon^2 E\sin\theta}\delta^i{}_4 e^{-iE(x^0 + x^5\cos\theta)}. \tag{63}$$

Thus, an incident ϕ-field in the $x^4 - x^5$ plane causes string vibrations only in the x^4-direction (to first order).

Let us next consider the case when $\phi_{\rm in} = 0$ and $B^{\rm in}_{\mu\nu} = b_{\mu\nu} e^{ik_\rho x^\rho}$ with the same k_μ as in the previous case. This means that the field strength $H_{\rm in}$ becomes:

$$H^{\rm in}_{\mu\nu\rho} = i(k_\mu b_{\nu\rho} + k_\nu b_{\rho\mu} + k_\rho b_{\mu\nu}) e^{ik_\sigma x^\sigma}, \tag{64}$$

which when inserted into Eq. (59) yields

$$\varepsilon^2(\ddot{X}^{\rm Th}_i - \acute{\acute{X}}^{\rm Th}_i) = 2i(k_i b_{05} + k_0 b_{5i} + k_5 b_{i0}) e^{-iE(x^0 + x^5\cos\theta)}. \tag{65}$$

Using the ansatz (62) and the condition (39) we get

$$X^{\rm Th}_i = -\frac{2i}{\varepsilon^2 E \sin^2\theta} b_{i5} e^{-iE(x^0 + x^5\cos\theta)}. \tag{66}$$

Since we are treating the model in a linearized approximation, it is of course possible to consider a general superposition of several kinds of incoming plane waves.

5.3 Outgoing radiation

The next step in our solution of the scattering problem is to solve Eqs. (57) and (58). To do this, we need to find expressions for $\delta_{\rm Th}$ and $\Omega_{\rm Th}$. These are found by expanding Eq. (14) in orders of $\frac{\varepsilon^2}{\phi_\infty}$ by replacing X with $\frac{\varepsilon^2}{\phi_\infty} X_{\rm Th}$. Doing this, we see that

$$\begin{aligned}
\delta_{\rm Th} &= -\frac{1}{4!}\epsilon_{ijkl}\Big[X^{i_1}_{\rm Th}\partial_{i_1}\delta^{(4)}(x) dx^i\wedge dx^j\wedge dx^k\wedge dx^l + \\
&\quad + \delta^{(4)}(x)\Big(4\dot{X}^i_{\rm Th} dx^0\wedge dx^j\wedge dx^k\wedge dx^l + 4\acute{X}^i_{\rm Th} dx^5\wedge dx^j\wedge dx^k\wedge dx^l\Big)\Big] \qquad (67)\\
\Omega_{\rm Th} &= -X^i_{\rm Th}\partial_i\delta^{(4)}(x). \qquad (68)
\end{aligned}$$

Eq. (57) is hard to solve as it stands without breaking the self-duality condition, but we may decompose the self-dual $H_{\rm out}$ as

$$H_{\rm out} = H_{\rm el} + *H_{\rm el} \tag{69}$$

where $H_{\rm el}$ satisfies

$$\begin{aligned}
dH_{\rm el} &= 0 \qquad (70)\\
d^*H_{\rm el} &= 2\pi * \delta_{\rm Th}. \qquad (71)
\end{aligned}$$

This means that we can find a $B_{\rm el}$ such that $H_{\rm el} = dB_{\rm el}$. $B_{\rm el}$ should satisfy the same gauge fixing condition (36) as $B_{\rm out}$. In this gauge, the equations of motion for $\phi_{\rm out}$

and $B_{\rm el}$ are

$$
\begin{aligned}
\partial_\mu \partial^\mu \phi_{\rm out} &= 4\pi X^i_{\rm Th} \partial_i \delta^{(4)}(x) && (72)\\
\partial_\mu \partial^\mu B^{\rm el}_{ij} &= 0 && (73)\\
\partial_\mu \partial^\mu B^{\rm el}_{i5} &= 2\pi \dot{X}^{\rm Th}_i \delta^{(4)}(x) && (74)\\
\partial_\mu \partial^\mu B^{\rm el}_{0i} &= -2\pi \acute{X}^{\rm Th}_i \delta^{(4)}(x) && (75)\\
\partial_\mu \partial^\mu B^{\rm el}_{05} &= 2\pi X^i_{\rm Th} \partial_i \delta^{(4)}(x) && (76)
\end{aligned}
$$

To solve these, we need to find the Green function satisfying

$$
\partial_\mu \partial^\mu D(t, \mathbf{x}; t', \mathbf{x}') = \delta(t - t')\delta^{(4)}(x - x')\delta(x^5 - x'^5), \tag{77}
$$

where obviously $t \equiv x^0$ and $\mathbf{x} = (x^1, x^2, x^3, x^4, x^5)$. The retarded solution to this equation is (cf. [**?**], Ch. 12)

$$
\begin{aligned}
D_{\rm ret}(t, \mathbf{x}; t', \mathbf{x}') &= -\frac{1}{8\pi^2}\theta(t - t')\Big[\frac{1}{|\mathbf{x} - \mathbf{x}'|^3}\delta(t - t' - |\mathbf{x} - \mathbf{x}'|) + \\
&\quad + \frac{1}{|\mathbf{x} - \mathbf{x}'|^2}\delta'(t - t' - |\mathbf{x} - \mathbf{x}'|)\Big], && (78)
\end{aligned}
$$

where θ is the ordinary Heaviside step function.

We need to calculate the convolutions of this Green function with both $\delta^{(4)}(x)$ and $\partial_i \delta^{(4)}(x)$ multiplied by the exponential factor in the expression for $X_{\rm Th}$ (it is the only part of $X_{\rm Th}$ with dependence on x). The integrals thus obtained are not analytically solvable, but we may solve them in the case when $r \equiv \sqrt{x^i x_i}$ is very large compared to the wavelength. This is equivalent to saying that $rE \gg 1$, i.e. we look in the far-field region.

We have that

$$
\begin{aligned}
f(t, \mathbf{x}) &= \int dt' d^5x' D_{\rm ret}(t, \mathbf{x}; t', \mathbf{x}')\delta^{(4)}(x')e^{-iE(x'^0 + x'^5 \cos\theta)} && (79)\\
&= -\frac{1}{8\pi^2}\int dx'^5 \Big[\frac{1}{(r^2 + (x^5 - x'^5)^2)^{3/2}} - \\
&\quad - \frac{iE}{r^2 + (x^5 - x'^5)^2}\Big] e^{-iE(t - (r^2 + (x^5 - x'^5)^2)^{1/2} + x'^5 \cos\theta)}, && (80)
\end{aligned}
$$

which is as far as we get without doing approximations. The geometry of the present situation is shown in Fig. 2. The angle θ in this figure must be the same as the one in Fig. 1 since both k_5 and $k_\mu k^\mu$ are unaffected by the scattering. Note though that the plane in Fig. 2 does not have to be $x^4 - x^5$, the vector r may in fact point in any direction perpendicular to the string. This means that the different possible outgoing

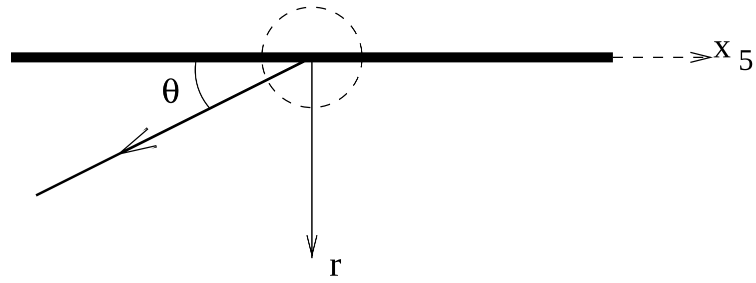

Figure 2: Outgoing wave in the r-x^5 plane.

wave vectors span a three-sphere. The dominating contribution to the field at a point x^μ is from the part of the string inside the dashed circle. This means that if we write

$$x'^5 = x^5 + \frac{r}{\tan\theta} + s\sqrt{\frac{r}{E}}, \tag{81}$$

then only small s-values should contribute significantly. Performing this substitution, we may expand the exponential and the denominators in the integral (80). The result of this substitution and the subsequent Taylor expansions is

$$f(t,\mathbf{x}) = -\frac{1}{8\pi^2}e^{-iE(x^0-r\sin\theta+x^5\cos\theta)}\int ds\sqrt{\frac{r}{E}}\frac{-iE}{r^2}\sin^2\theta e^{\frac{i}{2}\sin^3\theta s^2}\Big[1+\mathcal{O}\Big(\frac{1}{\sqrt{rE}}\Big)\Big] \tag{82}$$

which becomes, after performing the Gaussian integral,

$$f(t,\mathbf{x}) = \sqrt{\frac{E\sin\theta}{32\pi^3 r^3}}ie^{i\frac{\pi}{4}}e^{-iE(x^0-r\sin\theta+x^5\cos\theta)}\Big[1+\mathcal{O}\Big(\frac{1}{\sqrt{rE}}\Big)\Big]. \tag{83}$$

In the same way, we may calculate

$$g_i(t,\mathbf{x}) = \int dt' d^5x' D_{\mathrm{ret}}(t,\mathbf{x};t',\mathbf{x}')\partial'_i\delta^{(4)}(x')e^{-iE(x'^0+x'^5\cos\theta)}, \tag{84}$$

which yields

$$g_i(t,\mathbf{x}) = -\frac{x_i}{r}\sqrt{\frac{E^3\sin^3\theta}{32\pi^3 r^3}}e^{i\frac{\pi}{4}}e^{-iE(x^0-r\sin\theta+x^5\cos\theta)}\Big[1+\mathcal{O}\Big(\frac{1}{\sqrt{rE}}\Big)\Big]. \tag{85}$$

If we use the results just obtained together with the equations of motion for ϕ_{out} and B_{el} we get, to lowest order, the outgoing waves (we are only interested in the particular solutions, not the homogeneous)

$$\phi_{\mathrm{out}} = 4\pi\xi^i g_i \tag{86}$$
$$B^{\mathrm{el}}_{ij} = 0 \tag{87}$$
$$B^{\mathrm{el}}_{i5} = -2\pi iE\xi_i f \tag{88}$$
$$B^{\mathrm{el}}_{0i} = 2\pi iE\cos\theta\xi_i f \tag{89}$$
$$B^{\mathrm{el}}_{05} = 2\pi\xi^i g_i, \tag{90}$$

where ξ^i is defined in Eq. (62). Having found $B_{\rm el}$ we can easily find $H_{\rm out} = dB_{\rm el} + *dB_{\rm el}$. Defining the amplitude $h'_{\mu\nu\rho}$ by

$$\frac{\varepsilon^2}{\phi_\infty} H^{\rm out}_{\mu\nu\rho} \equiv h'_{\mu\nu\rho} e^{-iE(x^0 - r\sin\theta + x^5\cos\theta)}, \tag{91}$$

we get, after a small calculation, that

$$\begin{aligned}
h'_{ijk} &= \frac{\varepsilon^2}{\phi_\infty}\sqrt{\frac{E^5\sin^3\theta}{8\pi r^3}}\, i e^{i\frac{\pi}{4}} \sin\theta \epsilon_{ijki_1}\left(\xi^{i_1} - \frac{x^{i_1}x^{i_2}\xi_{i_2}}{r^2}\right) && (92)\\
h'_{0ij} &= \frac{\varepsilon^2}{\phi_\infty}\sqrt{\frac{E^5\sin^3\theta}{8\pi r^3}}\, i e^{i\frac{\pi}{4}} \left(\cos\theta \frac{x_i\xi_j - x_j\xi_i}{r} - \epsilon_{ij}{}^{kl}\frac{x_k\xi_l}{r}\right) && (93)\\
h'_{5ij} &= \frac{\varepsilon^2}{\phi_\infty}\sqrt{\frac{E^5\sin^3\theta}{8\pi r^3}}\, i e^{i\frac{\pi}{4}} \left(\frac{x_i\xi_j - x_j\xi_i}{r} - \cos\theta \epsilon_{ij}{}^{kl}\frac{x_k\xi_l}{r}\right) && (94)\\
h'_{05i} &= \frac{\varepsilon^2}{\phi_\infty}\sqrt{\frac{E^5\sin^3\theta}{8\pi r^3}}\, i e^{i\frac{\pi}{4}} \sin\theta\left(\xi_i - \frac{x_i x_{i_1}\xi^{i_1}}{r^2}\right). && (95)
\end{aligned}$$

Analogously, we define the amplitude ϕ'_0 by

$$\frac{\varepsilon^2}{\phi_\infty}\phi_{\rm out} = \phi'_0 e^{-iE(x^0 - r\sin\theta + x^5\cos\theta)}. \tag{96}$$

Eq. (86) then yields that

$$\phi'_0 = -\frac{\varepsilon^2}{\phi_\infty}\sqrt{\frac{E^3\sin^3\theta}{2\pi r^3}}\frac{x_i\xi^i}{r} e^{i\frac{\pi}{4}}. \tag{97}$$

We see that the fields decrease as $r^{-3/2}$ (at least, depending on direction). This makes sense, since it means that the energy decreases as r^{-3}, which is consistent with the fact that the possible outgoing wave vectors span a three-sphere. The wave vector k'_μ of the outgoing wave is

$$\begin{aligned}
k'_0 &= -E && (98)\\
k'_i &= E\sin\theta\frac{x_i}{r} && (99)\\
k'_5 &= -E\cos\theta, && (100)
\end{aligned}$$

meaning that the wave vector indeed points as in Fig. 2.

Far away from the string, $H_{\rm out}$ can be viewed as a plane wave, the only important x-dependence is then in the exponent. The amplitude factor may be taken to be constant. This means that $dH_{\rm out} = 0$; thus we can find a $B_{\rm out}$ such that $H_{\rm out} = dB_{\rm out}$. If we write

$$\frac{\varepsilon^2}{\phi_\infty} B^{\rm out}_{\mu\nu} = b'_{\mu\nu} e^{-iE(x^0 - r\sin\theta + x^5\cos\theta)}, \tag{101}$$

the apparent $b'_{\mu\nu}$ that yields the correct $H_{\rm out}$ is then given by

$$\begin{aligned} b'_{ij} &= \frac{\varepsilon^2}{\phi_\infty}\sqrt{\frac{E^3\sin^3\theta}{8\pi r^3}} e^{i\frac{\pi}{4}} \left(\epsilon_{ijkl}\frac{x^k\xi^l}{r} - \frac{x_i\xi_j - x_j\xi_i}{r}\cos\theta\right) && (102)\\ b'_{0i} &= 0 && (103)\\ b'_{i5} &= \frac{\varepsilon^2}{\phi_\infty}\sqrt{\frac{E^3\sin^3\theta}{8\pi r^3}} e^{i\frac{\pi}{4}} \sin\theta\left(\xi_i - \frac{x_ix_j\xi^j}{r^2}\right) && (104)\\ b'_{05} &= 0. && (105) \end{aligned}$$

It is also clear that this $B_{\rm out}$ satisfies the gauge fixing conditions (37) and (39) *if* considered as a plane wave.

5.4 Presenting the results

The results of the scattering problem may be presented in a more transparent way by exploiting the symmetry of the problem. When choosing the plane spanned by the incident wave vector and the string, the original $SO(5)$ symmetry is broken down to $SO(3)$. It is therefore convenient to introduce a three-dimensional vector space spanned by

$$b_a \equiv \frac{2}{\sin^2\theta} b_{ab} n^b, \tag{106}$$

where n is a unit vector pointing in the string direction and the prefactor is determined by requiring that

$$b^a b_a = b_{ab} b^{ab}. \tag{107}$$

The angle θ is still taken to be the angle between the incident wave and the string, which in terms of n and k is given by

$$\cos\theta = -n^a \hat{k}_a, \tag{108}$$

where the hat denotes a unit vector. Analogously, we define the angle φ as

$$\cos\varphi = \hat{k}^a \hat{k}'_a. \tag{109}$$

Obviously, the vector b is orthogonal to n and k, hence it has only three independent components (polarizations).

The inverse to Eq. (106) is given by

$$b_{ab} = b_{[a} n_{b]} + \cos\theta\, b_{[a}\hat{k}_{b]} + \frac{1}{2}\epsilon_{ab}{}^{cde} b_c n_d \hat{k}_e. \tag{110}$$

In our choice of coordinates, $n^a = (0,0,0,0,1)$, meaning that $b_5 = 0$. Thus we may omit this component and work only with

$$b_i = \frac{2}{\sin^2\theta} b_{i5}. \tag{111}$$

After these preliminaries, we may now express the scattered fields in terms of the incident in the following way:

$$\begin{aligned} b'_i &= A_{ij} b^j + C_i \phi_0 & (112)\\ \phi'_0 &= E_j b^j + F\phi_0, & (113)\end{aligned}$$

where b'_i is defined from b'_{ab} in the same way as b_i is defined from b_{ab}. The plus-sign in these relations should be interpreted with caution. The contributions from B and ϕ can be added like this *only* if they have the same wave vector k (and therefore the same energy). If this is not the case, they can still be added (since the problem is linear) but one must take into account that ϕ and B have different wave vectors.

We will now determine the coefficients A, C, E and F in these relations, which is done by using Eqs. (63), (66), (97) and (104). Note that we may always add any multiple of $\hat{k}_j$ to A_{ij} and E_j without altering Eqs. (112) and (113). In order to get unambiguous expressions, we require that $A_{ij}\hat{k}^j = E_j\hat{k}^j = 0$. This yields that

$$\begin{aligned}
A_{ij} &= \frac{1}{\phi_\infty}\sqrt{\frac{E}{2\pi r^3\sin^3\theta}}\, ie^{i\frac{\pi}{4}}\Big(-\sin^2\theta\delta_{ij} + \hat{k}'_i\hat{k}'_j + \hat{k}_i\hat{k}_j - \frac{\cos\varphi-\cos^2\theta}{\sin^2\theta}\hat{k}'_i\hat{k}_j\Big) & (114)\\
C_i &= \frac{1}{\phi_\infty}\sqrt{\frac{E}{2\pi r^3\sin^3\theta}}\, ie^{i\frac{\pi}{4}}\sin\theta\Big(\hat{k}_i - \frac{\cos\varphi-\cos^2\theta}{\sin^2\theta}\hat{k}'_i\Big) & (115)\\
E_i &= \frac{1}{\phi_\infty}\sqrt{\frac{E}{2\pi r^3\sin^3\theta}}\, ie^{i\frac{\pi}{4}}\sin\theta\Big(\hat{k}'_i - \frac{\cos\varphi-\cos^2\theta}{\sin^2\theta}\hat{k}_i\Big) & (116)\\
F &= \frac{1}{\phi_\infty}\sqrt{\frac{E}{2\pi r^3\sin^3\theta}}\, ie^{i\frac{\pi}{4}}\Big(\cos^2\theta - \cos\varphi\Big). & (117)
\end{aligned}$$

We see that the coefficients C_i and D_i are closely related, and that the matrix A_{ij} has a certain symmetry property. This means that the roles of incoming and outgoing waves may be interchanged, just as one would expect.

Acknowledgments: M.H. is a Research Fellow at the Royal Swedish Academy of Sciences. We would like to thank Martin Cederwall for discussions.